数理解析研究所講究録
第 1871 巻 2013 年 125-135

FINITE QUADRATIC MODULES OVER NUMBER FIELDS AND THEIR ASSOCIATED WEIL REPRESENTATIONS

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ABSTRACT. In this survey we report about recent research results in the theory of Weil representations of the Hilbert modular groups (and of their two-fold central extensions) associated the finite quadratic modules. We shall also indicate applications of these results to the theory of Jacobi forms over number fields.

1. INTRODUCTION

In the study of Hilbert, Jacobi and orthogonal modular forms of low weight over number fields it is essential to understand the representations of Hilbert modular groups or of their two-fold central extensions. The representations that are interesting in this context are called congruence representations.

Definition. *Congruence representations* are those complex representations of $\mathrm{SL}_2(\mathfrak{o})$ ($\mathfrak{o}$ the ring of integers in a number field) which are finite dimensional, and whose kernel is a congruence subgroup.

Remark. If $\mathfrak{o}$ is the ring of integers of $K \neq \mathbb{Q}$, and K not totally complex, then every subgroup of finite index in $\mathrm{SL}(2,\mathfrak{o})$ is a congruence subgroup [Ser70, Thm. 2, Cor. 3]. In particular, for such K a congruence representation is nothing else than a representation with finite image.

Let us consider, first of all,the case of $\mathrm{SL}(2,\mathbb{Z})$. For $K = \mathbb{Q}$, the key to the study of the congruence representations of $\mathrm{SL}(2,\mathbb{Z})$ are the Weil representations associated to finite quadratic modules. This is due to the following fact:

Theorem 1. [NW76] *Every congruence representation of* $\mathrm{SL}(2,\mathbb{Z})$ *is contained in a Weil representation associated to a finite quadratic module.* [1]

Knowing the congruence representations of $\mathrm{SL}(2,\mathbb{Z})$ gives rise to several applications e.g.

2010 *Mathematics Subject Classification.* 11F27 Primary, 11F50 Secondary.

[1]In [NW76] this theorem is not literally stated as given here. However, it is not difficult to deduce our statement from [NW76].

- determining all singular Jacobi forms over $\mathbb{Q}$ (for scalar index see [Sko85], and for arbitrary lattice index see [BS13c]),
- determining all Jacobi forms of critical weight over $\mathbb{Q}$ (see the article [BS13c]),
- proving vanishing results for Siegel modular forms of critical weight of degree 2. There are no Siegel modular forms of degree 2 on $\Gamma_0(N)$ of weight one [IS07],
- determining orthogonal modular forms of critical weight with signature $(2, n)$ (this is still an open project, critical weight is here $\frac{n-1}{2}$).

Recall that Jacobi's theta function is defined as

$$\begin{aligned}\vartheta(\tau, z) &= \sum_{r\in\mathbb{Z}} \left(\tfrac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{r}{2}} \\ &= q^{\frac{1}{8}}\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) \prod_{n>0} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1}) \\ &(q = e^{2\pi i\tau}, \zeta = e^{2\pi iz} \text{for } \tau \in \mathbb{H},\ z \in \mathbb{C}).\end{aligned}$$

The second identity is known as the Jacobi's triple product identity. (Here $\mathbb{H}$ denotes the upper half plane.) There is also another interesting function which can be written as a quotient of ϑ. Namely,

$$\vartheta^*(\tau, z) = \sum_{r\in\mathbb{Z}} \left(\tfrac{12}{r}\right) q^{\frac{r^2}{24}} \zeta^{\frac{r}{2}} = \frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)}.$$

We know that ϑ^* equals the Watson quintiple product identity, i.e. ϑ^* equals

$$\prod_{n\geq 1}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1})(1-z^2q^{2n-1})(1-z^{-2}q^{2n-1}).$$

The funtions ϑ and ϑ^* have η^3 and η as the first Taylor coefficients, i.e.

$$\vartheta(\tau, z) = \eta^3 z + O(z^3), \quad \vartheta^*(\tau, z) = \eta + O(z^2).$$

Here η is the Dedekind's eta function $\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$.

The functions ϑ and ϑ^* are important since ϑ is the Weierstrass σ-function and $\vartheta(\tau, z)$, for fixed τ, is the building block for all theta functions on the elliptic curve $E_\tau := \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. If E_τ is defined over $\mathbb{Q}$, then $\vartheta(\tau, z)$ is the contribution at infinity of the canonical height on E_τ. Moreover, ϑ and ϑ^* occur in the Jacobi triple and Watson quintuple product formulas, and these formulas have connections with Weyl-Kac denominator formulas for certain Kac-Moody algebras.

These two interesting functions can be characterized as the only singular Jacobi forms over $\mathbb{Q}$. Informally, Jacobi forms can be characterized as follows

Definition. For a half integer k, a positive $\mathbb{Z}$-lattice $\underline{L} = (L, \beta)$, an integer a mod 24, $J_{k,\underline{L}}(\varepsilon^a)$ is the space of holomorphic functions $\phi(\tau, z)$ of $\tau \in \mathbb{H}$ and $z \in \mathbb{C} \otimes_{\mathbb{Z}} L$ such that

(i) For fixed τ, the function $z \mapsto \phi(\tau, z)$ defines a section of a certain line bundle of $\mathbb{C} \otimes_{\mathbb{Z}} L/(\tau \otimes L + 1 \otimes L)$.
(ii) For any pair of elements x, y in $\mathbb{Q} \otimes_{\mathbb{Z}} L$, the function $\phi(\tau, x\tau + y)\, e(\tau\beta(x,x)/2)$ defines an elliptic modular form on $\mathrm{SL}(2,\mathbb{Z})$ of weight k with character ε^a.

Here ε is the character of the non-trivial double cover $\mathrm{Mp}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{Z})$ afforded by η. For the formal definition we refer the reader to [BS13c].

The functions ϕ described in this definition are called *Jacobi forms* of weight k index $\underline{L}$ and character ε. The first weight k where we expect non-zero Jacobi forms is $n/2$, where $n = \operatorname{rank} L$. The Jacobi forms of index $\underline{L}$ and of this weight are called *singular.*

We can classify all Jacobi forms of singular weight and scalar index over $K = \mathbb{Q}$. Namely, we have

Theorem 2. [Sko85, p. 27]
(i) $\vartheta \in J_{1/2,\underline{\mathbb{Z}}}(\varepsilon^3)$, $\vartheta^* \in J_{1/2,\underline{\mathbb{Z}}(3)}(\varepsilon)$.
(ii) *The functions ϑ and ϑ^* are the only Jacobi forms (of scalar index) of weight $1/2$ (up to trivial transformations in the z variable).*

The following explains the link between the space of singular Jacobi forms and the Weil representations of $\mathrm{Mp}(2,\mathbb{Z})$. For any positive integral lattice $\underline{L}$ of rank n, one has

$$J_{\frac{n}{2},\underline{L}}(\varepsilon^a) \cong \begin{array}{l}\text{Space of invariants of the tensor}\\ \text{product of the Weil representa-}\\ \text{tion associated to the } \textit{discrimi-}\\ \textit{nant module} \text{ of } \underline{L}(-1) \text{ with } \mathbb{C}(\varepsilon^a).\end{array}$$

Here $\underline{L}(-1) = (L, -\beta)$ if $\underline{L} = (L, \beta)$, and $\mathbb{C}(\varepsilon^a)$ is the $\mathrm{Mp}(2,\mathbb{Z})$-module $\mathbb{C}$ with the $\mathrm{Mp}(2,\mathbb{Z})$-action $(\alpha, z) \mapsto \varepsilon^a(\alpha) \cdot z$.

There are various new results and developments in the theory of Jacobi forms of singular weight for arbitrary lattice index over $K = \mathbb{Q}$. These are all joint work with Nils-Peter Skoruppa and can be found in the preprint [BS13c].

- Complete classification of all singular weight Jacobi forms over $K = \mathbb{Q}$ whose index is a rank 2-lattice.
- Complete classification of all singular weight Jacobi forms over $K = \mathbb{Q}$ whose index is a maximal integral lattice.
- A concise theory of Jacobi forms whose index is an odd lattice and the associated "shadow" representations (a generalization of Weil representations to "include discriminant modules of odd lattices").

In analogy we developed in our thesis [Boy11] a theory of finite quadratic modules over arbitrary number fields , and their associated Weil representations, and a (complete) theory for Jacobi forms over totally real number fields, and we determined all singular Jacobi forms of lattice rank one over totally real number fields.

In this article we shall report about the main features of this new theory of finite quadratic modules and associated Weil representations over arbitrary number field K, about an interesting new phenomena arising in the general theory over arbitrary number fields, and we indicate applications to the explicit construction of automorphic forms over number fields.

For an arbitrary number field K with ring of integers $\mathfrak{o}$ it is not known whether every congruence representation is contained a Weil representation (as it is the case over $\mathbb{Q}$).

However, for linear characters of $\mathrm{SL}(2,\mathfrak{o})$ (K totally real) it seems to be true since there is evidence due to a recent result (see Theorem 3 below) which describes explicitly the linear characters of Hilbert modular groups, and the explicit construction of Weil representations containing these characters for totally real number fields (which comes essentially from the classification of singular Jacobi forms of index of rank one over totally real number fields (see [Boy11]).

We know from [BS13a] that the congruence linear characters (the linear characters whose kernel is a congruence group) of $\mathrm{SL}(2,\mathfrak{o})$ for an arbitrary Dedekind domain $\mathfrak{o}$ is given by

Theorem 3. *Let $\mathfrak{o}$ be a Dedekind domain. The group of congruence linear characters of* $\mathrm{SL}(2,\mathfrak{o})$ *is given by:*

$$\prod_{\mathfrak{p}} \langle \varepsilon_{\mathfrak{p}} \rangle \times \prod_{\mathfrak{q}\|2} \langle \varepsilon_{\mathfrak{q}^2} \rangle \times \prod_{\mathfrak{r}^2|2} \left(\langle \varepsilon_{\mathfrak{r}} \rangle \times \langle \varepsilon'_{\mathfrak{r}^2} \rangle \right)$$

where $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$ run through all prime ideals of $\mathfrak{o}$ such that $\mathfrak{o}/\mathfrak{p} = \mathbb{F}_3$, $\mathfrak{o}/\mathfrak{q} = \mathbb{F}_2$, $\mathfrak{o}/\mathfrak{r} = \mathbb{F}_2$, and such that $\mathfrak{q}^2$ does not divide 2 and $\mathfrak{r}^2$ divides 2. (Here, for $\mathfrak{a} = \mathfrak{p}, \mathfrak{q}^2, \mathfrak{r}$, we use $\varepsilon_{\mathfrak{a}} = \varepsilon_N \circ$ red. modulo $\mathfrak{a}$, where $N \in \{2,3,4\}$ is such that $\mathfrak{o}/\mathfrak{a} = \mathbb{Z}/N$, and ε_N is a certain linear character of $\mathrm{SL}(2,\mathbb{Z}/N)$. *Moreover, $\varepsilon'_{\mathfrak{r}^2} = \varepsilon'_4 \circ$ red. modulo $\mathfrak{r}^2$, where ε'_4 is a certain linear character of* $\mathrm{SL}(2,\mathbb{F}_2[t]/(t^2))$*).*

2. Finite quadratic modules

In the following K is an arbitrary number field with ring of integers $\mathfrak{o}$ and different $\mathfrak{d}$. In this section we shall cite several results from [Boy11], where the theory of finite quadratic modules over number was first introduced.

Definition. A *finite quadratic module over K* (shortly $\mathfrak{o}$-FQM) is a pair (M,Q), where M is a finite $\mathfrak{o}$-module, and where Q is a *non-degenerate quadratic form on M*, i.e. where $Q : M \to K/\mathfrak{d}^{-1}$ is a map which satisfies the following properties:

(i) For all $a \in \mathfrak{o}$ and $x \in M$ one has $Q(ax) = a^2 Q(x)$.

(ii) The map $B : M \times M \to K/\mathfrak{d}^{-1}$ defined by $B(x, y) := Q(x + y) - Q(x) - Q(y)$ is $\mathfrak{o}$-bilinear and symmetric.

(iii) B is non-degenerate, i.e. $B(x, M) = \{0\}$ if and only if $x = 0$.

We shall define some notions concerning $\mathfrak{o}$-FQM, which will be useful below for our considerations.

Definition. The *annihilator* of $\mathfrak{M} = (M, Q)$ is the ideal

$$\operatorname{ann}(M) := \{a \in \mathfrak{o} \mid aM = 0\}.$$

The *level* of $\mathfrak{M}$ is the ideal

$$\operatorname{level}(M) := \{a \in \mathfrak{o} \mid aQ = 0\}.$$

Remark. The annihilator and the level contain the same prime ideals.

Example (Discriminant modules). *Let $\underline{L} = (L, \beta)$ be an even $\mathfrak{o}$-lattice, i.e. L is a finitely generated torsion-free $\mathfrak{o}$-module and $\beta : L \times L \to \mathfrak{d}^{-1}$ is a finitely generated symmetric, non-degenerate $\mathfrak{o}$-bilinear form such that $\beta(x, x) \in 2\mathfrak{d}^{-1}$.*

The dual of L is

$$L^{\#} = \{y \in \mathbb{Q} \otimes L \mid \beta(y, L) \subseteq \mathfrak{d}^{-1}\}.$$

The discriminant module of $\underline{L}$ is

$$D_{\underline{L}} = (L^{\#}/L, x + L \mapsto \beta(x) + \mathfrak{d}^{-1}).$$

It is easy to see that $D_{\underline{L}}$ is an $\mathfrak{o}$-FQM.

Over $\mathbb{Z}$, every $\mathfrak{o}$-FQM can be written as a discriminant module of an even $\mathbb{Z}$-lattice. This fact is no longer true when we consider $\mathfrak{o}$-FQM over an arbitrary number field. The following provides a counter example.

Example. *Consider the number field $K = \mathbb{Q}(\sqrt{17})$. Then we have $\mathfrak{o} = \mathfrak{o}_K = \mathbb{Z}[\frac{1+\sqrt{17}}{2}]$ and $\mathfrak{d} = \sqrt{17}\mathfrak{o}$. We have $2\mathfrak{o} = \mathfrak{p}\mathfrak{p}'$, where $\mathfrak{p} = \pi\mathfrak{o}$ and $\mathfrak{p}' = \pi'\mathfrak{o}$ are two distinct principal prime ideals in $\mathfrak{o}$ (with $\pi = (5 + \sqrt{17})/2$ and $\pi' = (5 - \sqrt{17})/2$). Then the $\mathfrak{o}$-FQM $\mathfrak{M} = \big(\mathfrak{o}/\pi\mathfrak{o}, x + \pi\mathfrak{o} \mapsto \frac{x^2}{\sqrt{17}\pi^2} + \mathfrak{d}^{-1}\big)$ is not a discriminant module of an $\mathfrak{o}$-lattice. Indeed, if $\mathfrak{M}$ equaled the discriminant module of the even $\mathfrak{o}$-lattice $\underline{L}$, then $\operatorname{rank}(L) = \operatorname{rank}(\mathfrak{o}_{\mathfrak{p}} \otimes L) = \operatorname{rank}(\mathfrak{o}_{\mathfrak{p}'} \otimes L)$. But $\mathfrak{o}_{\mathfrak{p}'} \otimes L$ would have to be even unimodular, hence of even rank, whereas $\mathfrak{o}_{\mathfrak{p}} \otimes L$ would to have be the direct sum of a unimodular even lattice plus π times a unimodular lattice of rank 1, whence of odd rank.*

Definition. An $\mathfrak{o}$-FQM (M, Q) is called *cyclic*, if the $\mathfrak{o}$-module M is cyclic, i.e. if there exists $x \in M$ such that $M = \mathfrak{o}x$. Henceforth, a cyclic $\mathfrak{o}$-FQM is called $\mathfrak{o}$-CM.

Proposition 1. [Boy11, Thm. 1.1] *Let $\omega \in K^*$ and $\mathfrak{l}$ be the denominator of $\omega\mathfrak{d}$, assume that $(2, \mathfrak{l})^2 | \mathfrak{l}$. Then $(\mathfrak{o}/\mathfrak{a}, x + \mathfrak{a} \mapsto \omega x^2 + \mathfrak{d}^{-1})$, where $\mathfrak{a} = \mathfrak{l}(2, \mathfrak{l})^{-1}$ is an $\mathfrak{o}$-CM, and every $\mathfrak{o}$-CM is isomorphic to such a module.*

Remark. Cyclic modules will play an important role for generalizing Jacobi's theta functions $\vartheta(\tau, z)$ and $\vartheta^*(\tau, z)$ to arbitrary totally real number fields.

There are three operations in the category of finite quadratic $\mathfrak{o}$-modules: twisting, direct sums and quotients. The most important one for our considerations is "taking quotients". For that we need to define

Definition. An $\mathfrak{o}$-submodule U of $\underline{M} = (M, Q)$ is called *isotropic*, if Q vanishes on U.

Definition. Let U be an isotropic submodule of $\mathfrak{M}$. Then the $\mathfrak{o}$-FQM

$$\mathfrak{M}/U := (U^{\#}/U, \underline{Q})$$

is called the *quotient* of $\mathfrak{M}$ by the isotropic submodule U. Here $U^{\#} = \{x \in M \mid B(x, M) = 0\}$ is the dual of U, and $\underline{Q}(x + U) := Q(x)$.

3. Weil representations associated to $\mathfrak{o}$-FQM

Theorem 4. [Boy11, Thm. 2.7] *Let $\mathfrak{M} = (M, Q)$ be an $\mathfrak{o}$-FQM. There is a projective representation of* $\mathrm{SL}(2, \mathfrak{o})$ *on* $\mathbb{C}[M]$ *such that*

$$\begin{aligned} T_b \cdot e_x &= \mathbf{e}(bQ(x))e_x \\ S \cdot e_x &= \frac{\sigma(\mathfrak{M})}{\sqrt{|M|}} \sum_{y \in M} \mathbf{e}(-Q(x, y)), \end{aligned} \tag{1}$$

where $\sigma(\mathfrak{M}) = \frac{1}{\sqrt{|M|}} \sum_{x \in M} \mathbf{e}(-Q(x))^2$. *Here* $T_b = \left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]$ *($b \in \mathfrak{o}$) and* $S = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$.

Remark. The proof of this theorem is not at all obvious. One can either proceed by citing parts of Weil's original paper [Wei64] and putting them together, or (as the author did in [Boy11]) one can prove from scratch by viewing the operators associated to this projective representation as intertwiners of certain representations of the Heisenberg group associated to the finite quadratic module in question. But in both ways, the proof is quite long.

Remark. The matrices $T_b := \left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right]$ ($b \in \mathfrak{o}$) and $S = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$ generate $\mathrm{SL}(2, \mathfrak{o})$ [Vas72, First Thm.]. Hence, once we know that such a projective representation exits we know that it is unique.

Theorem 5. [BS12] *If the annihilator of $\mathfrak{M}$ is an odd ideal, then the (projective) Weil representation associated to $\mathfrak{M}$ is a true representation of* $\mathrm{SL}(2, \mathfrak{o})$.

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[2] We use $\mathbf{e}(x) = e^{2\pi i \, \mathrm{tr}_{K/\mathbb{Q}}(x)}$.

The projective Weil representation can not always be lifted to a true representation to the well-known double cover $\mathrm{Mp}(2,\mathfrak{o})$ of $\mathrm{SL}(2,\mathfrak{o})$ which occurs in the theory of Hilbert modular forms of half integral weight, as we shall explain now.

For a $\mathfrak{p}$-module $\mathfrak{M} = (M, Q)$ (an $\mathfrak{o}$-FQM such that the annihilator of $\mathfrak{M}$ is a $\mathfrak{p}$-power) and a in $\mathfrak{o}^*$, we set

$$\gamma(a) = \frac{1}{\sqrt{|M|}} \sum_{x \in M} \mathbf{e}(aQ(x)).$$

We call γ the *Weil index* of $\mathfrak{M}$.

Definition. An integral ideal $\mathfrak{p}$ is called *bad for* $\mathfrak{M} = (M, Q)$, if $a \mapsto \gamma(a)/\gamma(1)$ is not a character of $\mathfrak{o}^*$, where γ is the Weil index associated to the $\mathfrak{p}$-part of $\mathfrak{M}$ (i.e. the $\mathfrak{o}$-FQM $(M_{\mathfrak{p}}, Q|_{M_{\mathfrak{p}}})$, where $M_{\mathfrak{p}}$ is the $\mathfrak{o}$-submodule of elements of M which are killed by a $\mathfrak{p}$-power).

Recall that for a local field F, the Kubota cocyle of F is the map $\kappa_F : \mathrm{SL}(2, F) \times \mathrm{SL}(2, F) \to \{\pm 1\}$ defined by

$$\kappa_F(A, B) = \left(\tfrac{x(A)}{x(AB)}, \tfrac{x(B)}{x(AB)}\right)_F, \quad x\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{otherwise.} \end{cases}$$

Let $\kappa = \prod_{\mathfrak{p}|2,\, \mathfrak{p} \text{ bad for } \mathfrak{M}} \kappa_{\mathfrak{p}}$, where $\kappa_{\mathfrak{p}}$ is the Kubota cocyle of the completion $K_{\mathfrak{p}}$ of K at $\mathfrak{p}$, and let $G := [\mathrm{SL}(2,\mathfrak{o}), \kappa]$ denote the central extension of $\mathrm{SL}(2,\mathfrak{o})$ defined by the cocyle κ.

Theorem 6. [BS13b, Thm. 6.2] *Let $\mathfrak{M}$ be an $\mathfrak{o}$-FQM. Then $\mathfrak{M}$ is a G-module.*

We have $\mathrm{Mp}(2,\mathfrak{o}) \simeq [\mathrm{SL}(2,\mathfrak{o}), \prod_{\mathfrak{p}|2} \kappa_{\mathfrak{p}}]$(see [BS12]). This fact together with Theorem 6 show that unlike $K = \mathbb{Q}$, the projective representation (1) of $\mathrm{SL}(2,\mathfrak{o})$ can indeed not always be lifted to a true representation of $\mathrm{Mp}(2,\mathfrak{o})$.

We can decompose Weil representations using the so-called methods of embedding and intertwining with the orthogonal group. These methods were first introduced by [Klo46], and where extended in [Boy11] to the theory of Weil representations over number fields.

Definition. We write $W(\mathfrak{M})$ for the G-module $\mathbb{C}[M]$ with the G-action (1), where T_b and S have to be replaced by $(T_b, +1)$ and $(S, +1)$. We shall refer to $W(M)$ as the *Weil representation* associated to $\mathfrak{M}$. The Weil representation associated to an $\mathfrak{o}$-CM is called a *cyclic Weil representation.*

Lemma 1. *Let U be an isotropic submodule of $\mathfrak{M}$. The map*

$$\iota_U : W(\mathfrak{M}/U) \hookrightarrow W(\mathfrak{M}), \quad e_X \mapsto \sum_{y \in X} e_y$$

defines a G-linear embedding (i.e. an injective G-module homomorphism).

Definition. By $\mathrm{O}(\mathfrak{M})$ we denote the group of automorphisms of $\mathfrak{M}$, i.e. the group of $\mathfrak{o}$-module automorphisms of M leaving Q invariant.

Lemma 2. *The natural action of $\mathrm{O}(\mathfrak{M})$ on $\mathbb{C}[M]$ intertwines with the action of G.*

Definition. By $\langle \cdot \mid \cdot \rangle$, we denote the Hermitian scalar product on $W(\mathfrak{M})$ which is anti-linear in the second argument and which satisfies:

$$\langle e_x \mid e_y \rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Definition. We define the *new part* $W(\mathfrak{M})^{new}$ of $W(\mathfrak{M})$ as the orthogonal complement with respect to (2) of the subspace

$$\sum_{\substack{U \subseteq \mathfrak{M} \\ U \text{isotropic} \\ U \neq 0}} \iota_U W(\mathfrak{M}/U).$$

Theorem 7. [Boy11, Thm. 2.2] *We have the following decomposition of $W(\mathfrak{M})$ into G-submodules:*

$$W(\mathfrak{M}) = W(\mathfrak{M})^{new} \oplus \sum_{\substack{U \subseteq \mathfrak{M} \\ U \text{ isotropic} \\ U \neq 0}} \iota_U W(\mathfrak{M}/U)^{new}. \tag{3}$$

If $\mathfrak{M}$ contains only one maximal isotropic submodule, then the second sum in (3) is an orthogonal sum with respect to the scalar product (2).

The proof of the first part can be done by doing induction on the dimension of $W(\mathfrak{M})$.

The condition that there exists only one maximal isotropic submodule is not necessary for the decomposition in (3) to be direct as the subsequent example shows. However, this condition is also not superfluous as we shall show in the second example below.

Example. *We show that the sum (3) applied to the finite quadratic $\mathbb{Z}$-module $\mathfrak{N} := (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, Q)$, where $Q(x + 2\mathbb{Z}, y + 2\mathbb{Z}) = xy/2 + \mathbb{Z}$, is direct. The nonzero isotropic submodules of $\mathfrak{N}$ are $U_1 = \langle([0], [1])\rangle$, $U_2 = \langle([1], [0])\rangle$. (Here we use $[x] = x + 2\mathbb{Z}$.) Since $|U_i^{\#}| \cdot |U_i| = 4$ (which follows from* [Boy11, Prop. 1.7]*) the quotient modules $\mathfrak{N}/U_i$ are trivial, in particular, $W(\mathfrak{N}/U_i) = W(\mathfrak{N}/U_i)^{new}$. They are spanned by the vectors $e_{([0],[0])} + e_{([0],[1])}$ and $e_{([0],[0])} + e_{([1],[0])}$, respectively, which are obviously linearly independent. We thus have $W(\mathfrak{N}) = W(\mathfrak{N})^{new} \oplus \iota_{U_1} W(\mathfrak{N}/U_1)^{new} \oplus \iota_{U_2} W(\mathfrak{N}/U_2)^{new}$.*

Example. *Let* $\mathfrak{N}' := (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, Q')$, *where* Q' *denotes the quadratic form* $Q'(x + 2\mathbb{Z}, y + 2\mathbb{Z}) = (x^2 + xy + y^2)/2 + \mathbb{Z}$. *We show that the sum* (3) *applied to* $\mathfrak{M} := \mathfrak{N}' \oplus \mathfrak{N}$, *where* $\mathfrak{N}$ *is as in the previous example, is not direct. The nonzero isotropic submodules of* $\mathfrak{M}$ *are* $U_1 = \langle([0],[0]) \oplus ([0],[1])\rangle$, $U_2 = \langle([0],[0]) \oplus ([1],[0])\rangle$, $U_3 = \langle([1],[1]) \oplus ([1],[1])\rangle$, $U_4 = \langle([0],[1]) \oplus ([1],[1])\rangle$ *and* $U_5 = \langle([1],[0]) \oplus ([1],[1])\rangle$. *Note that, for all* i, U_i *is maximal. The order of* $\mathfrak{M}/U_i$ *equals* 4. *Since the* U_i *are maximal, the finite quadratic* $\mathbb{Z}$*-modules* $\mathfrak{M}/U_i$ *are anisotropic, i.e have no nonzero isotropic submodules. (In fact, one can show that* $\mathfrak{M}/U_i$ *is isomorphic to* $\mathfrak{N}'$.*) Hence we have* $\iota_{U_i} W(\mathfrak{M}/U_i) = \iota_{U_i} W(\mathfrak{M}/U_i)^{new}$. *Since* $W(\mathfrak{M})$ *has dimension* 16 *the sum of the five four-dimensional spaces* $\iota_{U_i} W(\mathfrak{M}/U_i)^{new}$ *cannot be direct.*

Theorem 8. [Boy11, Thm. 2.3] *For each irreducible character of the group* $\mathrm{O}(\mathfrak{M})$, *the sum the spaces* $W(\mathfrak{M})^{new,\chi}$ *of those* $\mathrm{O}(\mathfrak{M})$*-submodules of* $W(\mathfrak{M})^{new}$ *which afford the character* χ, *is invariant under* G. *In particular, we have the decomposition of* $W(\mathfrak{M})^{new}$ *into* G*-submodules*

$$W(\mathfrak{M})^{new} = \bigoplus_{\chi \in \widehat{\mathrm{O}(\mathfrak{M})}} W(\mathfrak{M})^{new,\chi}. \tag{4}$$

(Recall $\widehat{\mathrm{O}(\mathfrak{M})}$ *denotes the set of irreducible characters of the orthogonal group* $\mathrm{O}(\mathfrak{M})$.*)*

The proof uses standard facts from representation theory. The group $\mathrm{O}(\mathfrak{M})$ intertwines with the action of the representation.

If we confine ourselves to $\mathfrak{o}$-CM, we obtain in fact complete decompositions of Weil representations as the subsequent theorem shows:

Theorem 9. [Boy11, Thm. 2.4] *Let* $\mathfrak{M}$ *be an* $\mathfrak{o}$*-CM with level* $\mathfrak{l}$ *and annihilator* $\mathfrak{a}$. *We set* $\mathfrak{m} = \mathfrak{l}(2,\mathfrak{l})^{-2}$.

(i) *We have the decomposition of* $W(\mathfrak{M})$ *into* G*-submodules:*

$$W(\mathfrak{M}) = \bigoplus_{\mathfrak{b}^2 | \mathfrak{m}} \iota_{\mathfrak{a}\mathfrak{b}^{-1}M} W(\mathfrak{M}/\mathfrak{a}\mathfrak{b}^{-1}M)^{new}.$$

Here the sum is over all integral $\mathfrak{o}$*-ideals* $\mathfrak{b}$ *whose square divides* $\mathfrak{m}$.

(ii) *For* $W(\mathfrak{M})^{new}$ *we have the decomposition*

$$W(\mathfrak{M})^{new} = \bigoplus_{\substack{\mathfrak{f} | \mathfrak{m} \\ \mathfrak{f}\, square-free}} W(\mathfrak{M})^{new,\mathfrak{f}}$$

into G*-submodules. The spaces* $W(\mathfrak{M})^{new,\mathfrak{f}}$ *are irreducible* G*-submodules.*

(iii) *For any square-free divisor* $\mathfrak{f}$ *of* $\mathfrak{m}$, *the dimension of the space* $W(\mathfrak{M})^{new,\mathfrak{f}}$ *equals*

$$\mathrm{Norm}(\mathfrak{a}) \prod_{\mathfrak{p}\|\mathfrak{m}} \frac{1}{2}\left(1+\frac{\mu(\mathfrak{f},\mathfrak{p})}{\mathrm{Norm}(\mathfrak{p})}\right) \prod_{\mathfrak{p}^2|\mathfrak{m}} \frac{1}{2}\left(1-\frac{1}{\mathrm{Norm}(\mathfrak{p}^2)}\right).$$

Remark. The proof follows from the previously stated two theorems and an upper bound for the number of irreducible submodules of a Weil representation. The decomposition (3) is a direct sum for $\mathfrak{o}$-CM, since a cyclic $\mathfrak{M}$ contains only one maximal isotropic submodule (one can determine explicitly the isotropic submodules of an $\mathfrak{o}$-CM).

The components of the decomposition (4) are in general not irreducible G-modules. However, for $\mathfrak{o}$-CM, they are irreducible. Indeed, we count the number of components occurring in the decomposition (4) and we compare this number to the upper bound for the number of irreducible representations occurring in a cyclic Weil representation. As it turns out the upper bound is sharp for cyclic $\mathfrak{o}$-FQM.

The upper bound follows from a formula for the absolute values of the traces of the Weil representations, which drop out when viewing the operators defined by the Weil representations as intertwiners of certain representations of the Heisenberg group associated to cyclic $\mathfrak{o}$-FQM.

4. An Application to automorphic forms

One can introduce a theory of Jacobi forms over totally real number fields which exhibits a lot of similarities with the theory of Jacobi forms over $\mathbb{Q}$ of lattice index.

In particular, Jacobi forms of weight k and index of rank n correspond to vector valued Hilbert modular forms of weight $k - n/2$. Singular weight Jacobi forms ($k = n/2$) correspond to vector-valued Hilbert modular forms of weight 0. From this we see that singular weight Jacobi forms whose index is an even $\mathfrak{o}$-lattice $\underline{L}$ correspond to one-dimensional submodules of the Weil representation of $\mathrm{Mp}(2, \mathfrak{o})$ associated to the discriminant module of $\underline{L}(-1)$. For Jacobi forms of lattice rank 1 these Weil representations are cyclic Weil representations. Hence, our decomposition yields all singular Jacobi forms over totally real number fields with rank one index.

5. Future Work

There is also a work in progress in the theory of Jacobi forms over number fields. Namely,

- Determining the critical weight Jacobi Forms of rank one index over (totally real) number fields. This depends on a characterization of Hilbert modular forms of weight 1/2. In the narrow class number one case these are all theta series [AS08]. This is a generalization of a theorem of Serre-Stark. The key is again the study of decomposition of certain Weil representations and determining one dimensional subrepresentations.

- Dimension formulas for vector valued modular forms and Jacobi forms over number fields [SS13].

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